

On the Power of Integers and Conductors of Quadratic Fields

Nihal Bircan^{*} and Michael E. Pohst^{**}

December 3, 2012

^{*} Berlin University of Technology, Institute for Mathematics, MA 8 – 1, Strasse des
17. Juni 136 D-10623 Berlin, Germany
Çankırı Karatekin University, Department of Mathematics, TR 18100, Çankırı, Turkey
bircan@math.tu-berlin.de

^{**} Berlin University of Technology, Institute for Mathematics, MA 8 – 1, D-10623
Berlin, Germany
pohst@math.tu-berlin.de

Abstract

We consider the integers α of the quadratic field $\mathbb{Q}(\sqrt{d})$ where $d \in \mathbb{Z}$ is square-free and $d \equiv 1, 2, 3 \pmod{4}$. Let p be an odd prime. Using the embedding into $\text{GL}(2, \mathbb{Z})$ we obtain bounds for the first $v \in \mathbb{N}$ such that $\alpha^v \equiv 1 \pmod{p}$. For the conductor f , we then study the first integer $n = n(f)$ such that $\alpha^n \in \mathcal{O}_f$. We obtain bounds for $n(f)$ and for $n(fp^k)$. The most interesting case is that α is the fundamental unit of $\mathbb{Q}(\sqrt{d})$.

2010 **Mathematics Subject Classification** : 11R11, 11R04, 42C05

Keywords and phrases: Chebyshev polynomials, conductor, integer of quadratic field

1 Introduction

We study the quadratic field $\mathbb{Q}(\sqrt{d})$ where $d \in \mathbb{Z}$ is square-free. We write $d = 4q + r$ where $r = 1, 2, 3$. The algebraic integers α of $\mathbb{Q}(\sqrt{d})$ are given by

$$(1.1) \quad \alpha = \begin{cases} a + b\sqrt{d}, & a, b \in \mathbb{Z} & \text{if } r = 2, 3 \\ \frac{1}{2}(a + b\sqrt{d}), & a, b \in \mathbb{Z}, a + b \in 2\mathbb{Z} & \text{if } r = 1. \end{cases}$$

Throughout the paper we consider any quadratic field and its integers α of any norm $\neq 0$. Let p be an odd prime. First we study the problem to find small exponents n such that $\alpha^n \equiv 1 \pmod{p}$. We will extensively use Legendre symbols and all the congruences are modulo p unless otherwise stated.

We adapt the classical Chebyshev polynomials T_n and U_n (for more information see [MOS66] section 5.7, [AbSt72] chapter 22) by defining

$$(1.2) \quad t_n(x) = t_n(x; s) = 2s^{n/2}T_n\left(\frac{x}{2\sqrt{s}}\right),$$

$$(1.3) \quad u_n(x) = u_n(x; s) = s^{n/2}U_n\left(\frac{x}{2\sqrt{s}}\right)$$

for $n \in \mathbb{N}_0$ where s is the norm of an algebraic integer in the quadratic field. These are unimodular polynomials with integer coefficients. For technical reasons we use this modification of Chebyshev polynomials in order to treat the case $d \equiv 1 \pmod{4}$ together with $d \equiv 2, 3 \pmod{4}$. For the properties of this adapted polynomials see Section 6. Then we specialize the results in the paper [BiPom] about $\text{GL}(2, \mathbb{Z})$ to quadratic fields. For previous works on this subject see e.g. [De79], [Jar05], [Ki89].

In Section 2, we consider matrices $A \in \text{GL}(2, \mathbb{Z})$ and how the integers of any quadratic field $\mathbb{Q}(\sqrt{d})$ can be embedded in $\text{GL}(2, \mathbb{Z})$. We also prove that $\alpha^n \equiv 1 \pmod{p}$ holds if and only if $A^n \equiv I \pmod{p}$. In the next sections we consider the algebraic integers of $\text{Norm}(\alpha) \neq 0$ and $\text{Norm}(\alpha) = \pm 1$ respectively. In these sections we apply the results of [BiPom] to the quadratic field case. In Section 5, for a given conductor f , we give upper estimates for

$$n(f) := \min\{v \in \mathbb{N} : \alpha^v \in \mathcal{O}_f\}$$

and $n(fp^k)$ for $k \in \mathbb{N}$ and the odd prime p .

2 The Embedding of Algebraic Integers of $\mathbb{Q}(\sqrt{d})$

Let $A \in \text{GL}(2, \mathbb{C})$, that is

$$(2.1) \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad a, b, c, d \in \mathbb{C}, \quad ad - bc \neq 0.$$

We always write

$$(2.2) \quad x := \text{tr } A = a + d, \quad s := \det A = ad - bc.$$

Proposition 2.1. *For $n \in \mathbb{N}$ we have*

$$(2.3) \quad A^n = u_{n-1}(x)A - su_{n-2}(x)I,$$

$$(2.4) \quad A^n = \frac{1}{2}t_n(x)I + u_{n-1}(x)(A - \frac{1}{2}xI).$$

This proposition is known in various forms. For instance, (2.3) with $s = 1$ is Lemma 3.1.3 in [MaRe03] where $p_n = u_{n-1}$ and $q_n = u_{n-2}$. The last matrix in (2.4) has zero trace and it follows that

$$(2.5) \quad \text{tr } A^n = t_n(x).$$

With the notation (2.1) we can write (2.4) as

$$(2.6) \quad A^n = \begin{pmatrix} \frac{1}{2}t_n(x) + \frac{1}{2}(a-d)u_{n-1}(x) & bu_{n-1}(x) \\ cu_{n-1}(x) & \frac{1}{2}t_n(x) - \frac{1}{2}(a-d)u_{n-1}(x) \end{pmatrix}.$$

Now, we consider the algebraic integers α of $\mathbb{Q}(\sqrt{d})$ using the notation (1.1). We define a homomorphism φ of the multiplicative group of integers $\alpha \neq 0$ into $GL(2, \mathbb{Z})$. For $r = 2, 3$ we set (see e.g. [Ba03, p. 38])

$$(2.7) \quad \varphi(\alpha) := A = \begin{pmatrix} a & b \\ bd & a \end{pmatrix}$$

whereas for $r = 1$ we set

$$(2.8) \quad \varphi(\alpha) := A = \begin{pmatrix} \frac{1}{2}(a+b) & b \\ qb & \frac{1}{2}(a-b) \end{pmatrix}.$$

It can be checked that this indeed defines an injective homomorphism. We have

$$(2.9) \quad s = \det A = \text{Norm}(\alpha) = \begin{cases} a^2 - b^2d & \text{if } r = 2, 3 \\ \frac{1}{4}(a^2 - b^2d) & \text{if } r = 1, \end{cases}$$

$$(2.10) \quad x = \text{tr } A = \begin{cases} 2a & \text{if } r = 2, 3 \\ a & \text{if } r = 1. \end{cases}$$

Since $A^n = \varphi(\alpha^n)$ and φ is injective, it follows from (2.6) that

$$(2.11) \quad \alpha^n = \begin{cases} \frac{1}{2}t_n(2a) + u_{n-1}(2a)b\sqrt{d} & \text{if } r = 2, 3 \\ \frac{1}{2}t_n(a) + \frac{1}{2}u_{n-1}(a)b\sqrt{d} & \text{if } r = 1. \end{cases}$$

Proposition 2.2. *If p is an odd prime and α_k, α_m are integers of $\mathbb{Q}(\sqrt{d})$ then $\alpha_k \equiv \alpha_m \pmod{p}$ if and only if $\varphi(\alpha_k) \equiv \varphi(\alpha_m) \pmod{p}$.*

Proof. We prove only the more complicated case $r = 1$. It can be proved in a similar way for $r = 2, 3$.

First we assume $\alpha_k \equiv \alpha_m \pmod{p}$ and we prove $\varphi(\alpha_k) \equiv \varphi(\alpha_m) \pmod{p}$. If $\alpha_k \equiv \alpha_m \pmod{p}$, this means

$$\alpha_k = \frac{1}{2}(a_k + b_k\sqrt{d}), \quad \alpha_m = \frac{1}{2}(a_m + b_m\sqrt{d}).$$

Hence, $a_k \equiv a_m$ and $b_k \equiv b_m \pmod{p}$. Then, we can write $a_k + b_k \equiv a_m + b_m$ and $a_k - b_k \equiv a_m - b_m \pmod{p}$. From congruence properties we can write

$$\frac{1}{2}(a_k + b_k) \equiv \frac{1}{2}(a_m + b_m), \quad \frac{1}{2}(a_k - b_k) \equiv \frac{1}{2}(a_m - b_m) \pmod{p}.$$

Then from (2.8), $\varphi(\alpha_k) \equiv \varphi(\alpha_m) \pmod{p}$.

Now we assume $\varphi(\alpha_k) \equiv \varphi(\alpha_m) \pmod{p}$ and prove $\alpha_k \equiv \alpha_m \pmod{p}$. Using the definition in (2.8) we can write

$$\varphi(\alpha_k) := \begin{pmatrix} \frac{1}{2}(a_k + b_k) & b_k \\ qb_k & \frac{1}{2}(a_k - b_k) \end{pmatrix}$$

and similarly, $\varphi(\alpha_m)$. Then, for modulo p , $b_k \equiv b_m$, $\frac{1}{2}(a_k + b_k) \equiv \frac{1}{2}(a_m + b_m)$ and $\frac{1}{2}(a_k - b_k) \equiv \frac{1}{2}(a_m - b_m)$. We can obtain $a_k \equiv a_m$ and this means $\alpha_k \equiv \alpha_m$. \square

Proposition 2.3. *If $p \nmid b$, $p \nmid d$ then $\alpha^n \equiv 1 \pmod{p}$ if and only if $A^n \equiv I \pmod{p}$.*

Proof. \Rightarrow First, for modulo p , we assume $\alpha^n \equiv 1$. For $r = 2, 3$,

$$\alpha^n = \frac{1}{2}t_n(x) + u_{n-1}(x)b\sqrt{d} \equiv 1$$

with $p \nmid b$, $p \nmid d$ where x is defined in (2.10). Since $u_{n-1}(x) \equiv 0$ by (2.11), $\frac{1}{2}t_n(x) \equiv 1$. Hence, $A^n = \frac{1}{2}t_n(x)I + u_{n-1}(x)(A - \frac{1}{2}xI) \equiv I$. For $r = 1$ namely, $\alpha^n = \frac{1}{2}t_n(x) + \frac{1}{2}u_{n-1}(x)b\sqrt{d}$ proof is similar.

\Leftarrow We assume $A^n \equiv I \pmod{p}$. Namely, for modulo p ,

$$A^n = \frac{1}{2}t_n(x)I + u_{n-1}(x)(A - \frac{1}{2}xI) \equiv I$$

and we aim to prove $\alpha^n = \frac{1}{2}t_n(x) + u_{n-1}(x)b\sqrt{d} \equiv 1$ for $r = 2, 3$. Since by (2.6) $bu_{n-1}(x) \equiv 0 \pmod{p}$, and as $b \not\equiv 0$, thus $u_{n-1}(x)(A - \frac{1}{2}xI) \equiv 0$ and $\text{tr}(A - \frac{1}{2}xI) \equiv 0$ this means

$$u_{n-1}(x) \begin{pmatrix} * & b \\ bd & * \end{pmatrix} \equiv 0.$$

Thus, $u_{n-1}(x)b \equiv 0$. According to our assumption $b \not\equiv 0 \pmod{p}$ and $u_{n-1}(x) \equiv 0 \pmod{p}$. Therefore from (2.6), $\frac{1}{2}t_n(x) \equiv 1 \pmod{p}$, for the cases $r = 2, 3$ and $r = 1$, $\alpha^n \equiv 1 \pmod{p}$. \square

3 Algebraic Integers with Norm $\neq 0$

In this section, we specialize the results of [BiPom] to the quadratic field case, using the embedding discussed in Section 2. We allow d to be negative. Again we write $d = 4q + r$ and $s = \text{Norm}(\alpha)$ where α is an integer of $\mathbb{Q}(\sqrt{d})$ as in (1.1).

Let p be an odd prime. All the following congruences will be modulo p . In the next theorem, α is any algebraic integer with $\text{Norm}(\alpha) \neq 0$ of the form (1.1). We assume that $p \nmid d$, $p \nmid b$ and

$$(3.1) \quad a^2 - 4s \not\equiv 0 \text{ if } r = 2, 3, \quad a^2 - s \not\equiv 0 \text{ if } r = 1.$$

Throughout the paper let x be the trace defined by (2.10) in the quadratic field case and s be the norm of α . Since t_n and u_n are polynomials with integer coefficients

the identities in Section 6 will become valid congruences. We always define ℓ as the Legendre symbol

$$(3.2) \quad \ell := \left(\frac{x^2 - 4s}{p} \right)$$

and write $p - \ell$ which is thus $p \mp 1$ for $\ell = \pm 1$.

Theorem 3.1. *Let p be an odd prime with $p \nmid d$, $p \nmid b$ and $s = N(\alpha) \neq 0$. Let ℓ be the Legendre symbol defined above. We set $\sigma = 1$ for $\ell = +1$ and $\sigma = s$ for $\ell = -1$. Then*

$$t_{p-\ell}(x) = 2\sigma, \quad u_{p-\ell-1}(x) \equiv 0.$$

We sum up the further results in the following table.

	$r = 2, 3$	$r = 1$
$\left(\frac{s}{p}\right) = +1$	$t_{\frac{p-\ell}{2}}(2a)^2 \equiv 4\sigma,$ $u_{\frac{p-\ell}{2}-1}(2a) \equiv 0$	$t_{\frac{p-\ell}{2}}(a)^2 \equiv 4\sigma,$ $u_{\frac{p-\ell}{2}-1}(a) \equiv 0$
$\left(\frac{s}{p}\right) = -1$	$t_{\frac{p-\ell}{2}}(2a) \equiv 0,$ $(a^2 - 4s)u_{\frac{p-\ell}{2}-1}(2a)^2 \equiv 4\sigma$	$t_{\frac{p-\ell}{2}}(a) \equiv 0,$ $(a^2 - s)u_{\frac{p-\ell}{2}-1}(a)^2 \equiv \sigma$

This is [BiPom, Th.4.1] specialized to our present situation.

The proof in [BiPom] used Chebyshev polynomials. In the present context of quadratic fields, many of the above formulas can be proved by other methods, see for instance [Ba03], [Le30, Th.1.7].

4 Algebraic Integers with Norm ± 1

First we consider the case that $s = \text{Norm}(\alpha) = +1$. All congruences will be modulo the odd prime p . Also ℓ is the Legendre symbol defined in (3.2) and x is defined in (2.10).

The following results are obtained by specializing the results in Section 5 and 6 of [BiPom]. The Legendre polynomials t_n and u_{n-1} depend only on x and s as defined in (2.9) and (2.10); the specific form (1.1) of α is not important.

Proposition 4.1. *Let $k \in \mathbb{N}$ divide $p - \ell$ and we assume that $\ell = \left(\frac{x^2 - 4s}{p}\right) \neq 0$. If $x \equiv t_k(y)$ for some $y \in \mathbb{Z}$ then, with $n = \frac{p-\ell}{k}$,*

$$(4.1) \quad t_n(x) \equiv 2, \quad u_{n-1}(x) \equiv 0, \quad \alpha^n \equiv 1.$$

See [BiPom, Th. 5.1].

For the special case that $k = 2^j$ we can say much more. We construct x_0, \dots, x_m recursively by the following rule. Let $x_0 = x$. If $\left(\frac{x+2}{p}\right) = -1$ then we set $m = 0$ and

stop. Now let $(\frac{x+2}{p}) = +1$ and suppose that x_0, \dots, x_k have already been constructed such that $2^k | p - \ell$ and

$$(4.2) \quad x_{v-1} \equiv t_2(x_v), ((x_v^2 - 4)/p) = \ell \quad \text{for } 1 \leq v \leq k.$$

If $2^{k+1} \nmid p - \ell$ or if $(\frac{x_k+2}{p}) = -1$ then we set $m = k$ and stop. Otherwise we have $2^{k+1} | p - \ell$ and $(\frac{x_k+2}{p}) = +1$. Then there exists x_{k+1} such that $x_k + 2 \equiv x_{k+1}^2$ and thus $x_k = t_2(x_{k+1})$. It follows from (4.2) that

$$((x_k - 2)/p) = ((x_k + 2)/p)((x_k - 2)/p) = ((x_k^2 - 4)/p) = \ell$$

and therefore $((x_{k+1}^2 - 4)/p) = ((x_k - 2)/p) = \ell$. This completes our construction. Note that $2^m | p - \ell$.

Theorem 4.2. *Let $N(\alpha) = 1$ and $\ell = (\frac{x^2-4}{p}) \neq 0$ and let x_0, \dots, x_m be as constructed above. Then*

$$(4.3) \quad t_{(p-\ell)/2^k}(x) \equiv 2 \quad \text{for } k = 0, \dots, m,$$

$$(4.4) \quad t_{(p-\ell)/2^{m+1}}(x) \equiv -2 \quad \text{or } 2^{m+1} \nmid p - \ell.$$

See [BiPom, Th. 4.3].

Corollary 4.3. *Let $s = N(\alpha) = 1$ and $\ell = (\frac{x^2-4}{p}) \neq 0$ and let x_0, \dots, x_m be constructed as above. Writing $n = (p - \ell)/2^m$ we have*

$$(4.5) \quad u_{n-1}(x) \equiv 0, \alpha^n \equiv 1.$$

If $2^{m+1} | p - \ell$ then furthermore

$$(4.6) \quad u_{\frac{n}{2}-1}(x) \equiv 0, \alpha^{n/2} \equiv -1.$$

These bounds are best possible: If $2^{m+2} | p - \ell$ then $u_{\frac{n}{2}-1}(x) \neq 0$.

Proof. Since $s = 1$ and for modulo p , $x^2 - 4 \not\equiv 0$ it follows from (6.1) and (4.3) that $u_{n-1} \equiv 0$ and therefore $A^n \equiv I$ by (2.4). By Proposition 2.3 we have $\alpha^n \equiv 1$. This proves (4.5). Furthermore if $2^{m+1} | p - \ell$ then (4.6) follows from (4.4) by the same argument. Now let $2^{m+2} | p - \ell$. Then it follows from (5.4) that $t_{n/2}(x) \equiv -2$ so that $t_{n/4}(x) \equiv 0$ by recursion formulas of $t_n(x)$ which is similar to $u_n(x)$ in Section 6. Hence $u_{\frac{n}{4}-1}(x) \neq 0$. \square

Now we consider the more complicated case that $N(\alpha) = -1$ so that in general $t_n(x) = t_n(x; -1)$. As before we set $\ell := (\frac{x^2-4s}{p})$ and assume that (3.1) with $s = -1$ holds. All congruences will be modulo the odd prime p . Since $(-1/p) = (-1)^{(p-1)/2}$ we obtain from Theorem 3.1 (where now $\sigma = \ell$) that, with $n = \frac{p-\ell}{2}$,

$$(4.7) \quad t_{2n}(x) \equiv 2\ell, \quad t_n(x)^2 \equiv 4\ell, \quad u_{n-1}(x) \equiv 0 \quad \text{for } p = 4q + 1,$$

$$(4.8) \quad t_{2n}(x) \equiv 2\ell, \quad t_n(x) \equiv 0, \quad u_{n-1}(x) \not\equiv 0 \quad \text{for } p = 4q + 3,$$

and it follows from (6.3) that

$$(4.9) \quad t_{2(p-\ell)}(x) \equiv 2.$$

Hence $t_n(x) \equiv \pm 2$ holds if and only if $p = 4q + 1$ and $\ell = +1$, which we now assume. It follows from (6.7) and $t_2(x; -1) = x^2 + 2$ that

$$(4.10) \quad t_{2n}(x; -1) = t_n(x^2 + 2; 1) \quad \text{for } n \in \mathbb{N}.$$

Since $(\frac{-1}{p}) = +1$ there exists $j \in \mathbb{Z}$ with $j^2 \equiv -1$. We now assume that $x \not\equiv 0$ and $x \not\equiv \pm 2j$. Then

$$(4.11) \quad (x^2 + 2)^2 - 4 = x^2(x^2 + 4) \not\equiv 0.$$

As in section 4, we construct numbers y_0, \dots, y_m with the only difference that $y_0 = x^2 + 2$ instead of $x_0 = x$. It follows from (4.11) that also $((y_0^2 - 4)/p) = \ell$. We have $y_0 + 2 = x^2 + 4$ and thus $((y_0 + 2)/p) = \ell = +1$. Hence the first step of our construction can be done so that $m \geq 1$. The construction stops if $((y_m + 2)/p) = -1$ or $2^{m+1} \nmid p - 1$.

Theorem 4.4. *Let $N(\alpha) = -1$, $p = 4q + 1$, $a^2 + 4 \not\equiv 0$, $\ell = +1$ and let y_0, \dots, y_m be constructed as above. Then $m \geq 1$ and*

$$(4.12) \quad t_{(p-1)/2^k}(x) \equiv 2 \quad \text{for } k = 0, \dots, m-1,$$

$$(4.13) \quad t_{(p-1)/2^m}(x) \equiv \begin{cases} -2, & \text{or} \\ 0 & \text{and } 2^{m+1} \nmid p - \ell. \end{cases}$$

See [BiPom, Th.6.1]. The next result is not a surprise because $N(\alpha^2) = 1$. Its proof is similar to the proof of Corollary 4.3.

Corollary 4.5. *Under the assumptions of Theorem 4.4, we now write $n = (p - \ell)/2^{m-1}$. Then (4.5) holds, and if $2^{m+1} \mid p - \ell$ then (4.6) also holds. These bounds are best possible: If $2^{m+1} \mid p - \ell$ then $u_{\frac{p-1}{4}-1}(x) \not\equiv 0$.*

Theorem 4.6. *Let $N(\alpha) = -1$ and let k be odd with $k \mid p - \ell$. If $x^2 + 4 \not\equiv 0$ and $x \equiv t_k(y; -1)$ for some $y \in \mathbb{Z}$ then, with $n = (p - 1)/k$,*

$$(4.14) \quad t_{2n}(x) \equiv 2, \quad t_n(x) \equiv 2\ell, \quad \alpha^n \equiv \ell.$$

Proof. Proof is given more generally in [BiPom]. □

5 Estimates for Conductors

We continue to study the quadratic field $\mathbb{Q}(\sqrt{d})$ where $d \equiv r = 1, 2, 3 \pmod{4}$. The order of the conductor $f \in \mathbb{N}$ is

$$(5.1) \quad \mathcal{O}_f = \begin{cases} \{a' + b'f\sqrt{d} : a', b' \in \mathbb{Z}\} & \text{if } r = 2, 3, \\ \{\frac{1}{2}(a' + (f-1)b') + \frac{1}{2}b'f\sqrt{d} : a', b' \in \mathbb{Z}, 2 \mid a' + b'\} & \text{if } r = 1. \end{cases}$$

We fix an integer α of $\mathbb{Q}(\sqrt{d})$ with $s = N(\alpha) \neq 0$ and x is given by (2.10); again we use the notation (1.1). The most interesting case is that α is the fundamental unit of $\mathbb{Q}(\sqrt{d})$. Following Halter-Koch we define

$$(5.2) \quad n(f) = n(f, \alpha) := \min\{v \in \mathbb{N} : \alpha^v \in \mathcal{O}_f\}.$$

Lemma 5.1. *Let $b \neq 0$ be given by (1.1) and s, x by (2.9). We write*

$$(5.3) \quad c := \gcd(b, f), b_0 := b/c, f_0 = f/c$$

Then we have

$$(5.4) \quad n(f) = n(f_0) = \min\{v \in \mathbb{N} : u_{v-1}(x; s) \equiv 0 \pmod{f_0}\}.$$

Proof. By (2.9) and (2.10) we have

$$\alpha^v \in \mathcal{O}_f \Leftrightarrow bu_{v-1}(x) \equiv 0 \pmod{f}.$$

Since $\gcd(b_0, f_0) = 1$ by (5.3), it follows that

$$\alpha^v \in \mathcal{O}_f \Leftrightarrow b_0 u_{v-1}(x) \equiv 0 \pmod{f_0} \Leftrightarrow u_{v-1}(x) \equiv 0 \pmod{f_0}.$$

Note that b has not been replaced by b_0 . Therefore we still have $u_{v-1}(x) = u_{v-1}(x; s)$ with unchanged x and s . \square

Let $g \in \mathbb{N}$ and $\gcd(b, g) = \gcd(f, g) = 1$. Then it follows from (5.4) and (6.5) that $u_{n(f)n(g)-1}(x; s) \equiv 0$ modulo f and also modulo g . Hence we have

$$(5.5) \quad n(fg) \leq n(f)n(g) \text{ if } \gcd(f, g) = 1.$$

For an odd prime p we define

$$(5.6) \quad q(p) = q(p; \alpha) := \min\{v \in \mathbb{N} : u_{v-1}(x; s) \equiv 0 \pmod{p}\}.$$

The results of Sections 3 and 4 provide upper estimates of $q(p)$. These results refer only on x and s and not explicitly on a, b and d in (1.1).

First let $\ell = \left(\frac{x^2-4s}{p}\right) \neq 0$. If $s = 1$ it follows from Corollary 4.2 that

$$q(p) \leq \frac{p-\ell}{2^m}, \text{ and } q(p) \leq \frac{p-\ell}{2^{m+1}} \text{ if } 2^{m+1} \mid p-\ell.$$

If $s = -1, p \equiv 1 \pmod{4}$ and $\ell = +1$ then it follows from Corollary 4.4 that

$$q(p) \leq \frac{p-\ell}{2^{m-1}}, \text{ and } q(p) \leq \frac{p-\ell}{2^m} \text{ if } 2^m \mid p-\ell.$$

Now let $x^2 - 4s \equiv 0 \pmod{p}$. Then it follows from (6.1) that $2^{v-1}u_{v-1}(x; s) \equiv vx^{v-1} \pmod{p}$. We conclude that $q(p) = p$ if $p \nmid s$ and $q(p) = 2$ if $p \mid s$.

Theorem 5.2. *If $\gcd(f, b) = 1$ and $p \nmid f$ then*

$$(5.7) \quad n(p^k f) \leq q(p)p^{k-1}n(f) \text{ for } k \geq 1.$$

Proof. We use induction on k . By (5.4) and (6.5) we have $u_{q(p)n(f)-1}(x; s) \equiv 0 \pmod{f}$, but also modulo p by (5.6) and (6.5). Since $\gcd(f, p) = 1$ it follows that the congruence is true also modulo pf . Hence (5.7) holds for $k = 1$ in view of (5.4).

Now let (5.7) hold for k . We write $v = q(p)p^{k-1}n(f)$ and have, by (5.7),

$$(5.8) \quad u_{v-1}(x; s) \equiv 0 \pmod{p^k f}.$$

We apply (6.1) with $n = p$ and with s^v instead of s . The binomial coefficients in the sum are divisible by the prime p . Since $2^{p-1} \equiv 1 \pmod{p}$ we thus obtain for $z \in \mathbb{Z}$ that

$$u_{p-1}(z; s^v) \equiv (z^2 - 4s^v)^{(p-1)/2} \pmod{p}.$$

For $z = t_v(x; s)$ we obtain by (6.2), that

$$(5.9) \quad u_{p-1}(t_v(x; s); s^v) \equiv \left[(x^2 - 4s)u_{v-1}(x; s) \right]^{\frac{p-1}{2}} \equiv 0 \pmod{p}$$

because of (5.8) where $k \geq 1$. Now we apply (6.4) with $m = p$ and $n = v$. By (5.8) and (5.9) we obtain

$$u_{q(p)p^{k-1}}(x; s) = u_{pv-1}(x; s) \equiv 0 \pmod{p^{k+1}f}.$$

Hence it follows from (5.4) that $n(p^{k+1}f) \leq q(p)p^k$. □

Theorem 5.3. *Let $f \in \mathbb{N}$ be odd and let f_0 be defined by (5.3). We write*

$$(5.10) \quad f_0 = \prod_{v=1}^{\mu} p_v^{k_v} \quad (k_v \in \mathbb{N})$$

with different primes p_v . Then

$$(5.11) \quad n(f) \leq \prod_{v=1}^{\mu} (q(p_v)p_v^{k_v-1}).$$

Proof. Let $g_0 = 1$ and, for $1 \leq \lambda \leq \mu$, let g_λ be the subproduct of (5.10) for $v = 1, \dots, \lambda$. Then $g_\lambda = p_\lambda^{k_\lambda} g_{\lambda-1}$ and $p_\lambda \nmid g_{\lambda-1}$. Hence we obtain from Theorem 5.2 applied to f_0 that

$$n(f_\lambda) \leq q(p_\lambda)p_\lambda^{k_\lambda-1}n(f_{\lambda-1}),$$

and (5.11) with f replaced by f_0 follows by induction. Finally we use that $n(f) = n(f_0)$ by Lemma 5.1. □

6 Some Formulas for Chebyshev Polynomials

We gather some formulas that we need to prove our results, concentrating on the polynomials u_n defined in (1.3). See [MOS66, Sect.5.7,] and [BiPom]. For odd n and $x, s \in \mathbb{C}$, we have

$$(6.1) \quad u_{n-1}(x; s) = \frac{1}{2^{n-1}} \sum_{k=0}^{(n-3)/2} \binom{n}{2k+1} x^{n-2k-1} (x^2 - 4s)^k + \frac{1}{2^{n-1}} (x^2 - 4s)^{\frac{n-1}{2}}.$$

The recursion formula $u_{n+1}(x) = xu_n(x) - su_{n-1}(x)$ shows that, for $x, s \in \mathbb{C}$,

$$\begin{aligned} u_0(x) &= 1, \quad u_1(x) = x, \quad u_2(x) = x^2 - s, \quad u_3(x) = x^3 - 2sx, \\ u_4(x) &= x^4 - 3sx^2 + s^2, \quad u_5(x) = x^5 - 4sx^3 + 2s^2x. \end{aligned}$$

Furthermore that $t_n(x; s), u_n(x; s) \in \mathbb{Z}[x, s]$. For $n \in \mathbb{N}$ we have

$$(6.2) \quad (x^2 - 4s)u_{n-1}(x; s)^2 = t_n(x; s)^2 - 4s^n,$$

$$(6.3) \quad t_n(x; s)^2 = t_{2n}(x; s) + 2s^n.$$

We need a relation for products which involves different parameters.

$$(6.4) \quad u_{mn-1}(x; s) = u_{m-1}(t_n(x; s); s^n)u_{n-1}(x; s) \quad (m, n \in \mathbb{N}).$$

It follows that, for $\mu \in \mathbb{N}$ and $x, s \in \mathbb{Z}$,

$$(6.5) \quad u_{n-1}(x; s) \equiv 0 \pmod{\mu} \Rightarrow u_{mn-1}(x; s) \equiv 0 \pmod{\mu}.$$

To prove (6.4) it is sufficient to consider $\frac{x}{2\sqrt{s}} = \cos \theta$ with real θ . Then it follows from (1.2), (1.3) and the properties [MOS66, p.257,] of the T_n and U_n that

$$(6.6) \quad t_n(x; s) = 2s^{\frac{n}{2}} \cos(n\theta), \quad u_{m-1}(x; s) = s^{\frac{m-1}{2}} \frac{\sin(m\theta)}{\sin \theta}.$$

By (1.3) and (1.2) we therefore have

$$\begin{aligned} u_{m-1}(t_n(x; s); s^n) &= s^{n\frac{m-1}{2}} U_{m-1}\left(\frac{1}{2s^{n/2}t_n(x; s)}\right) \\ &= s^{n\frac{m-1}{2}} U_{m-1}(\cos(n\theta)) = s^{\frac{mn-n}{2}} \frac{\sin(mn\theta)}{\sin n\theta}. \end{aligned}$$

Now we multiply by $u_{n-1}(x; s)$. Using (6.6) we obtain

$$u_{m-1}(t_n(x; s); s^n)u_{n-1}(x; s) = s^{\frac{mn-1}{2}} \frac{\sin(mn\theta)}{\sin n\theta} = u_{mn-1}(x; s)$$

using again (6.6).

In Section 4 we use the following relation between the polynomials $t_n(x; s)$ with different parameters s . If $s \neq 0$ and $m, n \in \mathbb{N}$ then

$$(6.7) \quad t_{mn}(x; s) = t_n(t_m(x; s); s^m);$$

indeed it follows from (1.2) and the composition property $T_{mn} = T_n \circ T_m$ that

$$\begin{aligned} t_{mn}(x; s) &= 2(s^m)^{n/2} T_n\left(T_m\left(\frac{x}{2\sqrt{s}}\right)\right) \\ &= t_n\left(\frac{1}{2\sqrt{s^m}} T_m\left(\frac{x}{2\sqrt{s}}\right); s^m\right) \end{aligned}$$

which implies (6.7) by (1.2).

Acknowledgements

The authors would like to thank Prof. Dr. F. Halter-Koch for suggesting the interesting problem of Section 5, Prof. Dr. Christian Pommerenke for the identity (6.4) and also Prof. Dr. Attila Pethő for his valuable comments and remarks.

References

- [AbSt72] M. Abramowitz and I. A. Stegun, Handbook of mathematical functions, Dover Publications, New York, 1972
- [BiPom] N. Bircan, C. Pommerenke, On Chebyshev polynomials and $GL(2, \mathbb{Z}/p\mathbb{Z})$, *Bull. Math. Soc. Sci. Math. Roumanie*, 2012, vol. 55(103), no. 4, 353 – 364
- [Ba03] E. J. Barbeau, Pell’s equation, Springer New York, 2003
- [De79] J. Denef, The Diophantine problem for polynomial rings of positive characteristic, Logic Colloquium 78, North-Holland Publishing Company, 1979
- [Jar05] J. H. Jaroma, On the rank of apparition of composite N in Lehmer sequences, *Nonlinear Analysis*, 2005, 63, e1081 – e1086
- [Ki89] P. Kiss, On rank of apparition of primes in Lucas sequences, *Publ. Math. Debrecen*, 1989, 36, 147 – 151
- [Le30] D. H. Lehmer, An extended theory of Lucas functions, *Ann. of Math.*, 1930, 31, 419 – 448
- [MaRe03] C. Maclachlan and A. W. Reid, The arithmetic of hyperbolic 3-manifolds, Springer-Verlag, New York, 2003
- [MOS66] W. Magnus, F. Oberhettinger and R. P. Soni, Formulas and theorems for the special functions of mathematical physics, Springer-Verlag Berlin, 1966